A QUANTIFIED TAUBERIAN THEOREM FOR SEQUENCES

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Abstract. The main result of this paper is a quantified version of Ingham's Tauberian theorem for bounded vector-valued sequences rather than functions. It gives an estimate on the rate of decay of such a sequence in terms of the behaviour of a certain boundary function, with the quality of the estimate depending on the degree of smoothness this boundary function is assumed to possess. The result is then used to give a new proof of the quantified Katznelson-Tzafriri theorem recently obtained in [21].

1. Introduction

One of the cornerstones in the asymptotic theory of operators is the Katznelson-Tzafriri theorem [13, Theorem 1], which states the following.

Theorem 1.1. *Let X be a complex Banach space and suppose that $T \in \mathcal{B}(X)$ is power-bounded. Then*

$$\lim_{n\to\infty} \|T^n(I-T)\| = 0 \tag{1.1}$$

if any only if $\sigma(T) \cap \mathbb{T} \subset \{1\}$.

Here $\mathcal{B}(X)$ denotes the algebra of bounded linear operators on a complex Banach space X, $\sigma(T)$ denotes the spectrum of the operator $T \in \mathcal{B}(X)$, and an operator $T \in \mathcal{B}(X)$ is said to be power-bounded if $\sup_{n\geq 0} \|T^n\| < \infty$. Moreover, $\mathbb{T}$ stands for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Limits of the type appearing in (1.1) play an important role for instance in the theory of iterative methods (see [16]), so it is natural to ask at what *speed* convergence takes place. If $\sigma(T) \cap \mathbb{T} = \emptyset$ the decay is at least exponential, with the rate determined by the spectral radius of T, so the real interest is in the non-trivial case where $\sigma(T) \cap \mathbb{T} = \{1\}$. Given a continuous non-increasing function $m : (0,\pi] \to [1,\infty)$ such that $\|R(\mathrm{e}^{\mathrm{i}\theta}, T)\| \leq m(|\theta|)$ for $0 < |\theta| \leq \pi$, it is shown in [21, Theorem 2.11] that, for any $c \in (0,1)$,

$$\|T^n(I-T)\| = O\big(m_{\log}^{-1}(cn)\big), \quad n \to \infty,$$

where $m_{\log}^{-1}$ is the inverse function of the map $m_{\log}$ defined by

$$m_{\log}(\varepsilon) = m(\varepsilon)\log\left(1 + \frac{m(\varepsilon)}{\varepsilon}\right), \quad 0 < \varepsilon \leq \pi, \tag{1.2}$$

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and where the statement $x_n = O(y_n)$, $n \to \infty$, for two sequences (x_n), (y_n) of non-negative terms, means that there exists a constant $C > 0$ such that $x_n \le C y_n$ for all sufficiently large $n \ge 0$. Moreover, this result is optimal in an important special case; see Remark 2.6(a) below.

The main new result of this paper, Theorem 2.1, is a Tauberian theorem for sequences. The result is formulated for bounded vector-valued sequences but, to the knowledge of the author, is new even in the scalar-valued case. It can be viewed as a discrete analogue of Ingham's classical Tauberian theorem for functions; however, it includes an estimate on the rate of decay. This is achieved by adapting a new technique developed recently in the setting of C_0-semigroups in [9] and going back to [3]. The result is then used, in Theorem 2.5, to give a new proof of the quantified version of Theorem 1.1 discussed above. For further related results in the general area may be found in [1], [2], [10], [13], [14], [17], [18], [19] and the references they contain.

2. Main results

2.1. **Preliminaries.** Let X be a complex Banach space and write $C_0(-\pi,\pi)$ for the set of continuous functions $\psi : [-\pi,\pi] \to \mathbb{C}$ which vanish in a neighbourhood of zero and satisfy $\psi(-\pi) = \psi(\pi)$. Further let $L^1_{\mathrm{loc}}(\mathbb{T}\backslash\{1\};X)$ denote the set of functions $F : \mathbb{T}\backslash\{1\} \to X$ such that the map $\theta \mapsto \psi(\theta)F(\mathrm{e}^{\mathrm{i}\theta})$, interpreted as taking the value zero when ψ does, lies in $L^1(-\pi,\pi;X)$ for all $\psi \in C_0(-\pi,\pi)$. Let $\mathbb{E} = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$, the exterior of the closed unit disc. Given a holomorphic function $G : \mathbb{E} \to X$ and given $F \in L^1_{\mathrm{loc}}(\mathbb{T}\backslash\{1\};X)$, F will be said to be a *boundary function* for G if

$$\lim_{r\to 1+} \int_{-\pi}^{\pi} \psi(\theta) G\big(r\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \psi(\theta) F\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta \tag{2.1}$$

for all $\psi \in C_0(-\pi,\pi)$. For $k \ge 1$, let $C^k(\mathbb{T}\backslash\{1\};X)$ denote the set of functions $F : \mathbb{T}\backslash\{1\} \to X$ which are k-times continuously differentiable, with $\mathbb{T}\backslash\{1\}$ viewed as a one-dimensional manifold, and let $C^\infty(\mathbb{T}\backslash\{1\};X) = \bigcap_{k\ge 1} C^k(\mathbb{T}\backslash\{1\};X)$.

2.2. **A quantified Tauberian theorem.** Theorem 2.1 below is the main result of this paper and can be viewed as a discrete analogue of Ingham's Tauberian theorem for functions; see [11] and also [12]. In the statement of the result, given $x \in \ell^\infty(\mathbb{Z}_+;X)$, $G_x : \mathbb{E} \to X$ denotes the holomorphic function given by

$$G_x(\lambda) = \sum_{n\ge 0} \frac{x_n}{\lambda^{n+1}}, \quad |\lambda| > 1.$$

The theorem shows that if $x \in \ell^\infty(\mathbb{Z}_+;X)$ has uniformly bounded partial sums and if G_x possesses a boundary function F_x, then $x \in c_0(\mathbb{Z}_+;X)$. Moreover, the result gives an estimate on the rate of decay of $\|x_n\|$ as $n \to \infty$, the quality of which depends on the smoothness and the rate of growth near the point 1 of F_x. The proof uses a technique which goes back to [3] and has been extended recently in [9]. One advantage of this approach over the contour integral method used to obtain [21, Theorem 2.11] is that it extends to the case in which F_x is only finitely often continuously differentiable.

Given a continuous non-increasing function $m : (0, \pi] \to [1, \infty)$ and $k \geq 1$, define the function $m_k : (0, \pi] \to (0, \infty)$ by

$$m_k(\varepsilon) = m(\varepsilon) \left(\frac{m(\varepsilon)}{\varepsilon} \right)^{1/k}, \tag{2.2}$$

noting that, for each $k \geq 1$, m_k maps bijectively onto its range.

Theorem 2.1. *Let X be a complex Banach space and let $x \in \ell^\infty(\mathbb{Z}_+; X)$ be such that*

$$\sup_{n \geq 0} \left\| \sum_{k=0}^{n} x_k \right\| < \infty. \tag{2.3}$$

If G_x admits a boundary function $F_x \in L^1_{\mathrm{loc}}(\mathbb{T} \backslash \{1\}; X)$, then $x \in c_0(\mathbb{Z}_+; X)$.

Moreover, given a continuous non-increasing function $m : (0, \pi] \to [1, \infty)$, the following hold.

(a) Suppose that $F_x \in C^k(\mathbb{T} \backslash \{1\}; X)$ for some $k \geq 1$ and that

$$\|F_x^{(j)}(\mathrm{e}^{\mathrm{i}\theta})\| \leq C|\theta|^{\ell - j} m(|\theta|)^{\ell+1}, \quad 0 < |\theta| \leq \pi,\ 0 \leq j \leq \ell \leq k, \tag{2.4}$$

for some constant $C > 0$. Then, for any $c > 0$,

$$\|x_n\| = O\left(m_k^{-1}(cn)\right), \quad n \to \infty, \tag{2.5}$$

where m_k^{-1} is the inverse function of the map m_k defined in (2.2).

(b) Suppose that $F_x \in C^\infty(\mathbb{T} \backslash \{1\}; X)$ and that

$$\|F_x^{(j)}(\mathrm{e}^{\mathrm{i}\theta})\| \leq Cj!|\theta| m(|\theta|)^{j+1}, \quad 0 < |\theta| \leq \pi,\ j \geq 0, \tag{2.6}$$

for some constant $C > 0$. Then, for any $c \in (0, 1)$,

$$\|x_n\| = O\left(m_{\log}^{-1}(cn) + \frac{1}{n}\right), \quad n \to \infty, \tag{2.7}$$

where $m_{\log}^{-1}$ is the inverse function of the map $m_{\log}$ defined in (1.2).

Remark 2.2. (a) Neither condition (2.3) nor the assumption that G_x admits a boundary function can be dropped, even in the scalar-valued case, as can be seen by considering the sequences $x = (1, 1, 1, \dots)$ and $x = (+1, -1, +1, -1, \dots)$, respectively.

(b) Note that if $m(\varepsilon) \geq c/\varepsilon$ for some $c > 0$, then (2.4) is satisfied if

$$\|F_x^{(j)}(\mathrm{e}^{\mathrm{i}\theta})\| \leq Cm(|\theta|)^{j+1} \quad 0 < |\theta| \leq \pi,\ 0 \leq j \leq k,$$

for some constant $C > 0$.

(c) Suppose that $m : (0, \pi] \to [1, \infty)$ is as in Theorem 2.1. If G_x has a holomorphic extension, denoted also by G_x, to a region containing

$$\Omega_{m,\theta} = \left\{ \lambda \in \mathbb{C} : |\lambda - \mathrm{e}^{\mathrm{i}\theta}| \leq \frac{1}{m(|\theta|)} \right\}$$

for $0 < |\theta| \leq \pi$ and if

$$\|G_x(\lambda)\| \leq C|\theta| m(|\theta|), \quad \lambda \in \Omega_{m,\theta},\ 0 < |\theta| \leq \pi,$$

for some constant $C > 0$, then a simple estimate using Cauchy's integral formula shows that (2.6) holds for the restriction F_x of G_x to $\mathbb{T} \backslash \{1\}$. This is analogous to the results for Laplace transforms in [4] and [15].

Conversely, if $F_x \in C^\infty(\mathbb{T}\backslash\{1\};X)$ and (2.6) holds, then F_x extends holomorphically to the region Ω_m given by

$$\Omega_m = \left\{\lambda \in \mathbb{C} : |\lambda - \mathrm{e}^{\mathrm{i}\theta}| < \frac{1}{m(|\theta|)},\ 0 < |\theta| \leq \pi\right\}.$$

Furthermore, if $G : \mathbb{E} \to X$ is a holomorphic function which admits a boundary function $F_x \in C^\infty(\mathbb{T}\backslash\{1\};X)$ satisfying (2.6), then G has a holomorphic extension which agrees with that of F_x on Ω_m. This follows from a standard Cayley transform argument combined with the 'edge-of-the-wedge theorem'; see for instance [20, §2 Theorem B].

Proof of Theorem 2.1. Let $\psi : [-\pi,\pi] \to \mathbb{R}$ be a smooth function such that $\psi(\theta) = 0$ for $|\theta| \leq 1$, $0 \leq \psi(\theta) \leq 1$ for $1 \leq |\theta| \leq 2$ and $\psi(\theta) = 1$ for $2 \leq |\theta| \leq \pi$. For $\varepsilon \in (0,\pi/2]$, let $\psi_\varepsilon, \varphi_\varepsilon : [-\pi,\pi] \to \mathbb{R}$ be given by $\psi_\varepsilon(\theta) = \psi(\theta/\varepsilon)$ and $\varphi_\varepsilon(\theta) = 1 - \psi_\varepsilon(\theta)$, $-\pi \leq \theta \leq \pi$. Moreover, for $n \in \mathbb{Z}$, let

$$y_n^\varepsilon = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta}\psi_\varepsilon(\theta)\,\mathrm{d}\theta \quad \text{and} \quad z_n^\varepsilon = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta}\varphi_\varepsilon(\theta)\,\mathrm{d}\theta.$$

Then $y_0^\varepsilon = 1 - z_0^\varepsilon$ and $y_n^\varepsilon = -z_n^\varepsilon$ for $n \neq 0$, and a simple calculation using integration by parts shows that $y^\varepsilon, z^\varepsilon \in \ell^1(\mathbb{Z})$. Let $x^\varepsilon \in \ell^\infty(\mathbb{Z};X)$ be given by $x^\varepsilon = x * y^\varepsilon$, so that $x_n^\varepsilon = \sum_{j\geq 0} x_j y_{n-j}^\varepsilon$ for $n \in \mathbb{Z}$. Then, setting $s_n = \sum_{j=0}^n x_j$ for $n \geq 0$,

$$x_n - x_n^\varepsilon = (x * z^\varepsilon)_n = \sum_{j\geq 0} s_j\big(z_{n-j}^\varepsilon - z_{n-j-1}^\varepsilon\big), \quad n \geq 0. \tag{2.8}$$

Since $\varphi_\varepsilon(\theta) = 0$ for $2\varepsilon \leq |\theta| \leq \pi$,

$$|z_n^\varepsilon - z_{n-1}^\varepsilon| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta}\big(1 - \mathrm{e}^{-\mathrm{i}\theta}\big)\varphi_\varepsilon(\theta)\,\mathrm{d}\theta\right| \lesssim \int_{-2\varepsilon}^{2\varepsilon} |\theta|\,\mathrm{d}\theta \lesssim \varepsilon^2 \tag{2.9}$$

for all $n \in \mathbb{Z}$. Here and in what follows the statement $p \lesssim q$ for real-valued quantities p and q means that $p \leq Cq$ for some number $C > 0$ which is independent of all the parameters that are free to vary, in this case of ε and n. Similarly, for $n \neq 0$, integrating by parts twice gives

$$|z_n^\varepsilon - z_{n-1}^\varepsilon| = \left|\frac{1}{2\pi n^2}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Big(\big(1 - \mathrm{e}^{-\mathrm{i}\theta}\big)\varphi_\varepsilon(\theta)\Big)\,\mathrm{d}\theta\right| \lesssim \frac{1}{n^2}. \tag{2.10}$$

For $n \geq 0$ and $\varepsilon \in (0,\pi/2]$, let $P_{n,\varepsilon} = \{j \geq 0 : |j - n| \leq \frac{1}{\varepsilon}\}$ and $Q_{n,\varepsilon} = \{j \geq 0 : |j - n| > \frac{1}{\varepsilon}\}$. Using (2.9) and (2.10) in (2.8), together with the fact that $s \in \ell^\infty(\mathbb{Z}_+;X)$ by assumption (2.3), it follows that

$$\|x_n - x_n^\varepsilon\| \lesssim \sum_{j\in P_{n,\varepsilon}} \varepsilon^2 + \sum_{j\in Q_{n,\varepsilon}} \frac{1}{(n-j)^2} \lesssim \varepsilon, \quad n \geq 0. \tag{2.11}$$

Now, by the dominated convergence theorem, Fubini's theorem and (2.1),

$$
\begin{aligned}
x_n^\varepsilon &= \lim_{r\to 1+} \sum_{j\geq 0} \frac{x_j}{r^{j+1}} y_{n-j}^\varepsilon \\
&= \lim_{r\to 1+} \frac{1}{2\pi} \sum_{j\geq 0} \int_{-\pi}^{\pi} \frac{x_j}{r^{j+1}} \mathrm{e}^{\mathrm{i}(n-j)\theta} \psi_\varepsilon(\theta)\,\mathrm{d}\theta \\
&= \lim_{r\to 1+} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n+1)\theta} \psi_\varepsilon(\theta) G_x\big(r\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n+1)\theta} \psi_\varepsilon(\theta) F_x\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta
\end{aligned} \tag{2.12}
$$

for all $n \in \mathbb{Z}$ and $\varepsilon \in (0, \pi/2]$. Hence $x^\varepsilon \in c_0(\mathbb{Z}; X)$ for each $\varepsilon \in (0, \pi/2]$ by the Riemann-Lebesgue lemma, and it follows from (2.11) that $x \in c_0(\mathbb{Z}_+; X)$.

Suppose $F_x \in C^k(\mathbb{T}\backslash\{1\}; X)$ for some $k \geq 1$. Integrating by parts k times in (2.12) and estimating crudely by means of (2.4) gives

$$
\|x_n^\varepsilon\| \lesssim \frac{1}{n^k} \sum_{j=0}^{k} m(\varepsilon)^{j+1} \lesssim \frac{m(\varepsilon)^{k+1}}{n^k} \tag{2.13}
$$

for all $n \geq 1$ and all $\varepsilon \in (0, \pi/2]$. Given $c > 0$ and $n \geq 1$ sufficiently large, let $\varepsilon_n \in (0, \pi/2]$ be given by $\varepsilon_n = m_k^{-1}(cn)$. The estimate (2.5) follows from (2.11) and (2.13) on setting on setting $\varepsilon = \varepsilon_n$ for sufficiently large $n \geq 1$.

Now suppose that $F_x \in C^\infty(\mathbb{T}\backslash\{1\}; X)$. In order to obtain the estimate (2.7), it is necessary to make explicit choices of the functions $\psi_\varepsilon, \varphi_\varepsilon : [-\pi, \pi] \to \mathbb{R}$ and hence of the sequences $y^\varepsilon, z^\varepsilon \in \ell^1(\mathbb{Z})$ of their Fourier coefficients. Thus, given $\varepsilon \in (0, \pi/2]$, let $y^\varepsilon \in \ell^1(\mathbb{Z})$ be given by $y_0^\varepsilon = 1 - \frac{3\varepsilon}{2\pi}$ and

$$
y_n^\varepsilon = \frac{\cos(2n\varepsilon) - \cos(n\varepsilon)}{\varepsilon\pi n^2}, \quad n \neq 0,
$$

and define $z^\varepsilon \in \ell^1(\mathbb{Z})$ by $z_0^\varepsilon = 1 - y_0^\varepsilon$ and $z_n^\varepsilon = -y_n^\varepsilon$ for $n \neq 0$. Moreover, let $x^\varepsilon \in \ell^\infty(\mathbb{Z}; X)$ be given by $x^\varepsilon = x * y^\varepsilon$, as before. Then the function

$$
\psi_\varepsilon(\theta) = \sum_{n\in\mathbb{Z}} \frac{y_n^\varepsilon}{\mathrm{e}^{\mathrm{i}n\theta}}, \quad -\pi \leq \theta \leq \pi,
$$

satisfies $\psi_\varepsilon(\theta) = 0$ for $|\theta| \leq \varepsilon$, $\psi_\varepsilon(\theta) = \varepsilon^{-1}|\theta| - 1$ for $\varepsilon \leq |\theta| \leq 2\varepsilon$, and $\psi_\varepsilon(\theta) = 1$ for $2\varepsilon \leq |\theta| \leq \pi$. Now (2.8) still holds but the above method for estimating $|z_n^\varepsilon - z_{n-1}^\varepsilon|$, $n \in \mathbb{Z}$, is no longer applicable since φ_ε is not differentiable. Instead, consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(0) = 0$ and

$$
\phi(t) = \frac{2}{\pi} \left(\frac{\cos(2t) - \cos(t)}{t^3} + \frac{\sin(2t) - \frac{1}{2}\sin(t)}{t^2} \right), \quad t \neq 0.
$$

Then $\phi \in L^1(\mathbb{R})$ and

$$
z_n^\varepsilon - z_{n-1}^\varepsilon = \varepsilon \int_{\varepsilon(n-1)}^{\varepsilon n} \phi(t)\,\mathrm{d}t, \quad n \in \mathbb{Z},
$$

and it follows from (2.3) and (2.8) that

$$
\|x_n - x_n^\varepsilon\| \leq \varepsilon \sum_{j\geq 0} \int_{\varepsilon(n-j-1)}^{\varepsilon(n-j)} |\phi(t)|\,\mathrm{d}t \lesssim \varepsilon, \quad n \geq 0. \tag{2.14}
$$

Now, by the same argument as in (2.12),

$$x_n^\varepsilon = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n+1)\theta}\psi_\varepsilon(\theta)F_x\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta, \quad n\in\mathbb{Z}.$$

Integrating by parts $k\geq 1$ times gives

$$\begin{aligned} x_n^\varepsilon &= A_{n,k}(-1)^k\int_{\varepsilon\leq|\theta|\leq\pi} \mathrm{e}^{\mathrm{i}(n+k+1)\theta}\psi_\varepsilon(\theta)F_x^{(k)}\big(\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{d}\theta\\ &\quad + B_{n,k}\frac{(-1)^k}{2\pi\varepsilon\mathrm{i}}\int_{\varepsilon\leq|\theta|\leq 2\varepsilon} \mathrm{e}^{\mathrm{i}(n+k)\theta}F_x^{(k-1)}\big(\mathrm{e}^{\mathrm{i}\theta}\big)\operatorname{sgn}\theta\,\mathrm{d}\theta\\ &\quad + \sum_{j=0}^{k-2} C_{n,j}\frac{(-1)^j}{2\pi\varepsilon}\Big[\mathrm{e}^{\mathrm{i}(n+j+1)\theta}F_x^{(j)}\big(\mathrm{e}^{\mathrm{i}\theta}\big)+\mathrm{e}^{-\mathrm{i}(n+j+1)\theta}F_x^{(j)}\big(\mathrm{e}^{-\mathrm{i}\theta}\big)\Big]_\varepsilon^{2\varepsilon} \end{aligned}$$

for all $n\geq 0$, where

$$A_{n,k}=\frac{n!}{(n+k)!},\quad B_{n,k}=\frac{n!}{(n+k-1)!}\sum_{j=1}^k\frac{1}{n+j}$$
$$\text{and}\quad C_{n,j}=\frac{n!}{(n+j+1)!}\sum_{\ell=1}^{j+1}\frac{1}{n+\ell}$$

for $0\leq j\leq k-2$. Now $A_{n,k}\leq n^{-k}$, $B_{n,k}\leq kn^{-k}$ and $C_{n,j}\leq (j+1)n^{-(j+2)}$ for $n\geq 1$. Thus (2.6) gives

$$\|x_n^\varepsilon\|\lesssim k!\frac{m(\varepsilon)^{k+1}}{n^k}+\frac{m(\varepsilon)}{n^2}\sum_{j=0}^{k-2}(j+1)!\frac{m(\varepsilon)^j}{n^j},\quad n,k\geq 1, \tag{2.15}$$

for $\varepsilon\in(0,\pi/2]$. Denote the two terms on the right-hand side of (2.15) by $D_{n,k}^\varepsilon$ and $E_{n,k}^\varepsilon$, respectively. For $c\in(0,1)$, Stirling's formula implies that $k!\lesssim(k/c\mathrm{e})^k$ for all $k\geq 0$ and hence

$$D_{n,k}^\varepsilon\lesssim m(\varepsilon)\left(\frac{km(\varepsilon)}{c\mathrm{e}n}\right)^k,\quad n,k\geq 1.$$

Let $k_{\varepsilon,n}=\lfloor cn/m(\varepsilon)\rfloor$. Then

$$D_{n,k_{\varepsilon,n}}^\varepsilon\lesssim m(\varepsilon)\exp\left(-\frac{cn}{m(\varepsilon)}\right) \tag{2.16}$$

for all $\varepsilon\in(0,\pi/2]$ and all $n\geq 1$ such that $k_{\varepsilon,n}\geq 1$. Moreover, for such values of ε and n, the choice of $k_{\varepsilon,n}$ ensures that

$$E_{n,k_{\varepsilon,n}}^\varepsilon\leq\frac{m(\varepsilon)}{n^2}\sum_{j=0}^{k-2}c^j\lesssim\frac{m(\varepsilon)}{n^2}. \tag{2.17}$$

Thus setting $k=k_{\varepsilon,n}$ in (2.15) and using (2.16) and (2.17) gives

$$\|x_n^\varepsilon\|\lesssim m(\varepsilon)\exp\left(-\frac{cn}{m(\varepsilon)}\right)+\frac{m(\varepsilon)}{n^2} \tag{2.18}$$

for all $\varepsilon \in (0, \pi/2]$ and $n \geq 1$ as above. Let $\varepsilon_n = m_{\log}^{-1}(cn)$ for $n \geq 1$ sufficiently large to ensure that $k_{\varepsilon_n,n} \geq 1$. For such values of n,

$$m(\varepsilon_n) \exp\left(-\frac{cn}{m(\varepsilon_n)}\right) \lesssim \varepsilon_n \quad \text{and} \quad \frac{m(\varepsilon_n)}{n^2} \lesssim \frac{1}{n},$$

so (2.7) follows from (2.14) and (2.18) on setting $\varepsilon = \varepsilon_n$. □

Remark 2.3. (a) The choice of $k_{\varepsilon,n}$ before equation (2.16) is motivated by the fact that, given any constant $C > 0$, the function $t \mapsto (Ct)^t$, defined on $(0, \infty)$, attains its global minimum at $t = (C\mathrm{e})^{-1}$.
(b) Theorem 2.1 can be extended to the case of a finite number of singularities on the unit circle; see also [15] and [21].

The following example illustrates that the estimates in Theorem 2.1 generally improve with the smoothness of the boundary function if $m(\varepsilon)$ grows moderately fast as $\varepsilon \to 0+$, but that the quality of these estimates can be independent of the degree of smoothness if this blow-up is very rapid.

Example 2.4. In Theorem 2.1, consider the function $m : (0, \pi] \to [1, \infty)$ given by $m(\varepsilon) = (\pi/\varepsilon)^\alpha$, where $\alpha \geq 1$. If $F_x \in C^k(\mathbb{T}\backslash\{1\}; X)$ for some $k \geq 1$ and (2.4) holds, then (2.5) gives

$$\|x_n\| = O\left(n^{-\frac{k}{\alpha(k+1)+1}}\right), \quad n \to \infty,$$

and, if $F_x \in C^\infty(\mathbb{T}\backslash\{1\}; X)$ and (2.6) holds, (2.7) becomes

$$\|x_n\| = O\left(\left(\frac{\log n}{n}\right)^{\frac{1}{\alpha}}\right), \quad n \to \infty. \tag{2.19}$$

Thus the estimate improves with the smoothness of F_x. By contrast, if the assumptions of Theorem 2.1 are satisfied for $m(\varepsilon) = \exp(\varepsilon^{-\alpha})$ with $\alpha > 0$, then (2.5) for any $k \geq 1$ and (2.7) all become

$$\|x_n\| = O\left((\log n)^{-\frac{1}{\alpha}}\right), \quad n \to \infty,$$

so in this case the quality of the estimate is unaffected by the smoothness of the boundary function.

2.3. **The quantified Katznelson-Tzafriri theorem.** The purpose of this final section is to deduce from Theorem 2.1 the quantified version of Theorem 1.1. This result was first obtained in [21] by means of a contour integral argument adapted from [4] and [15]. Some closely related results may be found for instance in [10], [14] and [17].

Theorem 2.5. *Let X be a complex Banach space and let $T \in \mathcal{B}(X)$ be a power-bounded operator such that $\sigma(T) \cap \mathbb{T} = \{1\}$. Suppose there exists a continuous non-increasing function $m : (0, \pi] \to [1, \infty)$ such that*

$$\|R(\mathrm{e}^{\mathrm{i}\theta}, T)\| \leq m(|\theta|), \quad 0 < |\theta| \leq \pi.$$

Then, for any $c \in (0, 1)$,

$$\|T^n(I - T)\| = O\big(m_{\log}^{-1}(cn)\big), \quad n \to \infty, \tag{2.20}$$

where $m_{\log}^{-1}$ is the inverse function of the map $m_{\log}$ defined in (1.2).

Proof. The result follows from part (b) of Theorem 2.1 applied, with X replaced by $\mathcal{B}(X)$, to the sequence x whose n-th term is $x_n = T^n(I-T)$, $n \geq 0$. Indeed, the sequence x is bounded since T is power-bounded, and moreover $\sum_{k=0}^n x_n = I - T^{n+1}$ for all $n \geq 0$, so (2.3) is also satisfied, again by power-boundedness of T. Furthermore, $G_x(\lambda) = (I-T)R(\lambda,T)$ for $|\lambda| > 1$. Let G_x denote also the extension of this map to the resolvent set $\rho(T) = \mathbb{C}\backslash\sigma(T)$ and let F_x be the restriction of G_x to $\mathbb{T}\backslash\{1\}$. Note that $\|R(\lambda,T)\| \geq \operatorname{dist}(\lambda,\sigma(T))^{-1} \geq |1-\lambda|^{-1}$ for all $\lambda \in \rho(T)$, and hence

$$m(\varepsilon) \geq \frac{1}{|1-\mathrm{e}^{\mathrm{i}\varepsilon}|} = \frac{1}{2\sin(\varepsilon/2)} \geq \frac{1}{\varepsilon}, \quad 0 < \varepsilon \leq \pi. \tag{2.21}$$

Further, for $k \geq 0$,

$$F_x^{(k)}(\lambda) = (-1)^k k! R(\lambda,T)^k \big(I + (1-\lambda)R(\lambda,T)\big), \quad \lambda \in \mathbb{T}\backslash\{1\},$$

and it follows from (2.21) that (2.6) holds. Since $m_{\log}(\varepsilon) \gtrsim m(\varepsilon) \geq \varepsilon^{-1}$ for all $\varepsilon \in (0,\pi]$, and therefore $n^{-1} \lesssim m_{\log}^{-1}(cn)$ for all sufficiently large $n \geq 1$, (2.20) follows from (2.7). $\square$

Remark 2.6. (a) For functions $m : (0,\pi] \to [1,\infty)$ of the form $m(\varepsilon) = C\varepsilon^{-\alpha}$ for suitable constants $C > 0$, as considered in the first part of Example 2.4, the right-hand side in (2.20) is given by that in (2.19). It is shown in [21, Section 3] that the logarithmic factor in this expression can be dropped if X is a Hilbert space but not for general Banach spaces.

(b) It is possible to obtain a 'local' version of Theorem 2.5 from Theorem 2.1 giving, for a fixed $x \in X$, an estimate for the rate of decay of $\|T^n(I-T)x\|$ as $n \to \infty$ which depends on the behaviour of the 'local' resolvent operator $R(\lambda,T)x$ as $|\lambda| \to 1+$; see for instance [5], [6], [7], [8] and [22] for related local results in the context mainly of C_0-semigroups. Similarly, Theorem 2.1 can be used to obtain an estimate on the rate of decay of weak orbits $\phi(T^n(I-T)x)$ as $n \to \infty$, where $x \in X$ and ϕ is a bounded linear functional on X.

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